Suppression of chaos by selective resonant parametric perturbations

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It is shown that, depending on its amplitude, frequency, and initial phase, a time-dependent periodic parametric perturbation can suppress chaos in nonlinear oscillators. The example of the Duffing-Holmes oscillator is used to demonstrate that all the numerically and experimentally observed phenomenology is theoretically explained by using the Melnikov-Holmes method and suppression of chaos is seen to be possible when certain resonant frequencies are involved.

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The problem of suppressing chaos has attracted great interest in recent years [1–4]. In particular, it has been observed both theoretically and experimentally [5–8] that resonant parametric perturbations [in the examples of the Josephson-junction model and the Duffing-Holmes (DH) equation] can suppress chaotic behavior arising from homoclinic bifurcations. In spite of this work, a complete comprehension of such inhibitory mechanisms is still far from being achieved. In fact, certain key questions remain. (i) What exactly is the nature of the resonant condition imposed on the parametric perturbations in order to regularize the dynamics? From numerical and experimental results, regularization is only observed for a *limited range* of resonances between the frequencies of the parametric perturbation and the primary chaos-inducing forcing. (ii) What is the influence of the initial phase difference between such forces. Experimentally it is found that [7]: "Indefinitely long regularization is found at exact resonance, but this also requires an appropriate phase relation between the forcing and the parametric perturbation." (iii) What is the nature of the route(s) from chaos to order underlying the inhibitory mechanism? (iv) What type of agreement might one expect between analytical and numerical results?

In this article I attempt to answer these questions using the example of the DH oscillator with a parametric perturbation of the cubic term [5,7,8]

$$\ddot{x}-x+\beta[1+\eta\cos(\Omega t+\varphi)]x^3=-\delta\dot{x}+\gamma\cos(\omega t)\ , \qquad (1)$$

where Ω, η, and φ are the frequency, amplitude, and initial phase, respectively, of the parametric perturbation ($\eta \ll 1$), which has a suppressory effect on the chaotic dynamics of the unperturbed system. By using the Melnikov-Holmes method (MHM) [9–13], analytical predictions are obtained for the threshold for chaos. In distinction to previous work [5,7,8], I obtain from these predictions a *selective* resonance condition involving ω, Ω, and φ, and *two* threshold values (upper and lower) for η for the elimination of the chaotic dynamics present at $\eta=0$. Numerical experiments are performed with Eq. (1) and the results are compared separately with the theoretically predicted values of Ω, η and φ that suppress chaos. Additionally, it is shown that type-II intermittency appears as Ω approaches the resonance condition.

As is well known, the MHM is only concerned with transient chaos [11], i.e., only *necessary* conditions for (steady) chaos are obtained from it, and therefore one always has the possibility of finding *sufficient* conditions for the elimination of (even transient) chaos. Observe that the validity of these statements is subject to two constraints since the MHM is a *perturbative* (to first order) method: (a) its predictions are only valid for motions based at points *sufficiently near* the separatrix of the unperturbed system; (b) the perturbative term's amplitude must be *sufficiently small* ($\ll 1$).

Let us now consider the concrete application of these ideas to system (1). From Cuadros and Chacón [8], the Melnikov function (MF) for this case ($\varphi \neq 0$) is written

$$M(t_0)=A(\gamma,\omega)\sin(\omega t_0)-B(\eta,\Omega)\sin(\Omega t_0+\varphi)-C(\beta,\delta)\ , \qquad (2)$$

with

$$A(\gamma,\omega)=\left[\frac{2}{\beta}\right]^{1/2}\pi\gamma\omega\,\mathrm{sech}\left[\frac{\pi\omega}{2}\right]\ ,$$
$$B(\eta,\Omega)=\frac{\pi\eta}{6\beta}(\Omega^4+4\Omega^2)\mathrm{csch}\left[\frac{\pi\Omega}{2}\right]\ , \qquad (3)$$
$$C(\beta,\delta)=\frac{4\delta}{3\beta}\ .$$

(The corresponding result for $\varphi=0$ from Lima and Pettini [Eq. (1), Ref. [8]] is incorrect.) Suppose that for $\eta=0$ we are in a chaotic situation for which the associated MF $M_0(t_0)=A(\gamma,\omega)\sin(\omega t_0)-C(\beta,\delta)$ changes sign at some t_0, i.e.,

$$A(\gamma,\omega)-C(\beta,\delta)\equiv d\geq 0\ , \qquad (4)$$

where the $\geq$ equals sign corresponds to the case of tangency between the stable and unstable manifolds.

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If we now let the parametric perturbation act on the system ($\eta \neq 0$) such that $B(\eta,\Omega)\leq d$, i.e., $A(\gamma,\omega)-B(\eta,\Omega)-C(\beta,\delta)\geq 0$, this equation represents a sufficient condition for $M(t_0)$ to change sign at some t_0. Thus, using Eq. (3), a necessary condition for $M(t_0)$ to always have the same sign, i.e., $M(t_0)<0$, is written

$$\eta > \left[1-\frac{C(\beta,\delta)}{A(\gamma,\omega)}\right] R(\gamma,\omega,\Omega) , \tag{5}$$

with

$$R(\gamma,\omega,\Omega)=\frac{6\sqrt{2\beta}\gamma\omega}{(\Omega^4+4\Omega^2)}\frac{\sinh\left[\frac{\pi\Omega}{2}\right]}{\cosh\left[\frac{\pi\omega}{2}\right]} . \tag{6}$$

For general Ω and φ, we shall see that this condition is not sufficient to ensure the negativity of $M(t_0)$. In order to obtain such a sufficient condition, we shall first need three lemmas.

Lemma I. Let Ω/ω be irrational. Then there is some $\bar{t}_0$ such $A(\gamma,\omega)\sin(\omega\bar{t}_0)-B(\eta,\Omega)\sin(\Omega\bar{t}_0+\varphi)>A(\gamma,\omega)-B(\eta,\Omega)$.

Lemma II. Let $q\Omega=p\omega$ for some positive integers p and q. Then a t_0^* exists such that $\sin(\omega t_0^*)=\sin(\Omega t_0^*+\varphi)=1$ if and only if

$$\frac{p}{q}=\frac{4m+1-2\varphi/\pi}{4n+1} \tag{7}$$

for some integers m and n.

Remark. Observe that a requirement is $\varphi=l_1\pi/l_2$, $l_{1,2}$ integers, for Eq. (7) to be fulfilled for some integers m and n. For the particular case ($\varphi=0$) considered in Refs. [5,8], Lemma II implies that, for general p and q, it is not always possible to find integers m,n fulfilling Eq. (7). Thus the condition given in those references is only a necessary (but *not sufficient)* condition for eliminating chaotic dynamics.

Lemma III. Let $f(t;p,q)\equiv[1-\cos(pt/q)]/(1-\cos t)$, t real, p and q integers. Then f is finite if and only if $q=1$. One also has that $0\leq f(t;p,1)\leq p^2$.

The proofs of these lemmas are quite straightforward, so they will be given elsewhere [14].

It is obvious that for Eq. (5) to be also a *sufficient* condition for $M(t_0)$ to be negative for all t_0, one must have

$$A(\gamma\omega)-B(\eta,\Omega)\geq A(\gamma,\omega)\sin(\omega t_0) -B(\eta,\Omega)\sin(\Omega t_0+\varphi) . \tag{8}$$

Now we look for the values of ω, Ω, and φ permitting Eq. (8) to be fulfilled for all t_0. From Lemma I, a resonance condition is required: $p\omega=q\Omega$. In such a situation, Lemma II provides a condition for Eq. (8) to be satisfied for a *infinity* of t_o values. Thus let us suppose that p, q, and φ verify Eq. (7). One can then rewrite Eq. (8) in the form

$$\frac{A(\gamma,\omega)}{B(\eta,\Omega)}\geq\frac{1-\cos(pt/q)}{[1-\cos(t)]} , \tag{9}$$

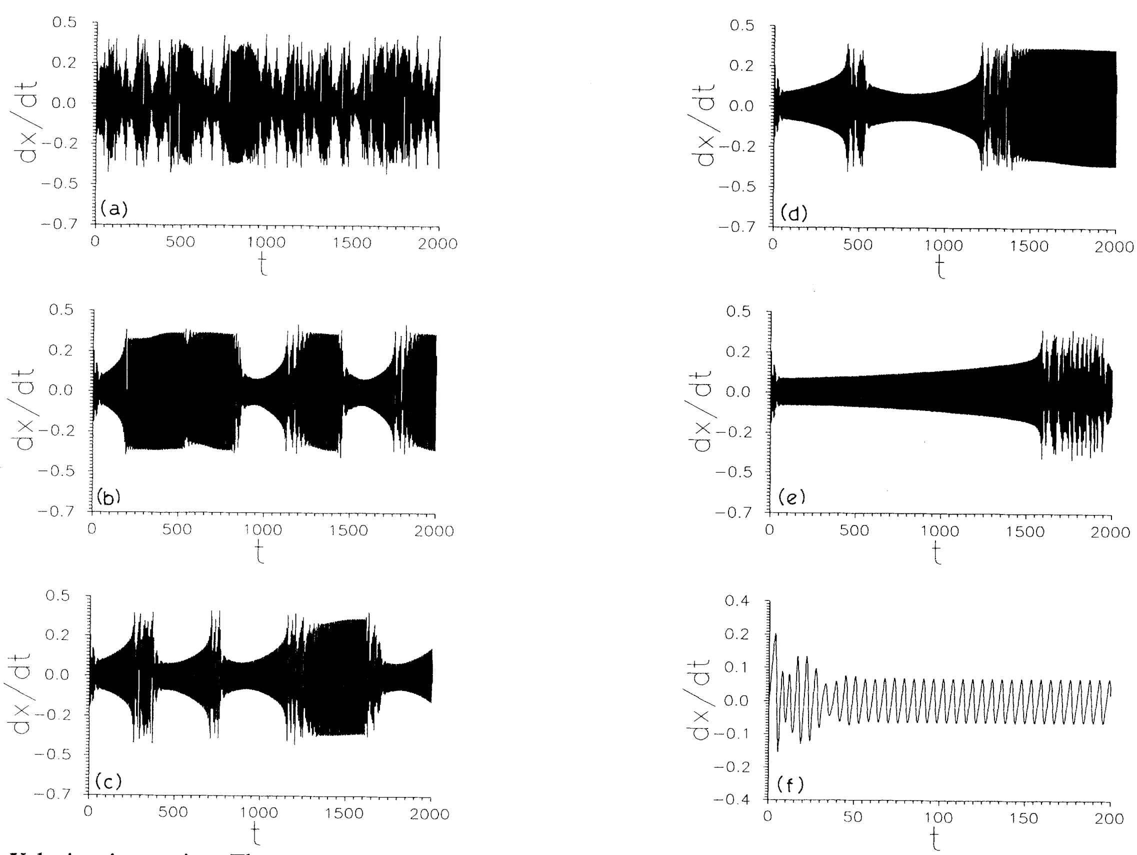

FIG. 1. Velocity time series. The parameters are $\beta=4$, $\delta=0.154$, $\gamma=0.088$, $\varphi=0$, $\eta=\eta_{\min}=0.075\,092\,5$, $\omega=1.1$. (a) $\Omega=1.15$, (b) $\Omega=1.11$, (c) $\Omega=1.107$, (d) $\Omega=1.104$, (e) $\Omega=1.101$, and (f) $\Omega\equiv\omega=1.1$.

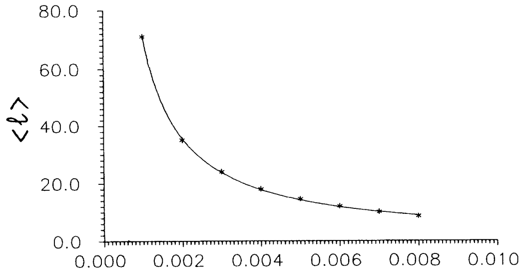

FIG. 2. Averaged length of laminar regions $\langle l \rangle$ versus $\varepsilon \equiv \Omega - 1.1$, for the same set of parameters as in Fig. 1.

with $t = \omega t_0 - (4n+1)\pi/2$. Finally, if $q=1$, Lemma III provides a condition for Eq. (9) to be fulfilled for all t:

$$\eta \le \frac{R(\gamma,\omega,\Omega)}{p^2}, \tag{10}$$

with $R(\gamma,\omega,\Omega)$ given by (6). In brief, we have the following.

Suppression theorem (ST). Let $\Omega = p\omega, p$ an integer, such that $p = (4m+1-2\varphi/\pi)/(4n+1)$ is satisfied for some integers m and n. Then $M(t_0)$ always has the same sign, i.e., $M(t_0) < 0$, if and only if the following condition in fulfilled:

$$\eta_{\min} < \eta \le \eta_{\max},$$

$$\eta_{\min} = \left[1 - \frac{C(\beta,\delta)}{A(\gamma,\omega)}\right] R(\gamma,\omega,\Omega), \tag{11}$$

$$\eta_{\max} = \frac{R(\gamma,\omega,\Omega)}{p^2}.$$

Remarks. First, observe that, for a given set of parameters satisfying the above theorem's hypothesis, as the resonance order p is increased, the allowed interval $]\eta_{\min}, \eta_{\max}]$ for suppression shrinks quickly. This permits one to explain why only a narrow range of resonances for suppressing chaos is observed in both numerical and real experiments [5,7,8]. Second, we can test the ST theoretically by considering the limiting case $\delta = 0$ (no damping). From Eq. (11), one has $\varphi = 2m\pi$ (m an integer), $\Omega = \omega$, and $\eta = R(\gamma,\omega,\Omega)$ as a sufficient and necessary condition for eliminating (Hamiltonian) chaos. [Note that this is the obvious result arising from a direct analysis of Eq. (2) with $\delta = 0$, i.e., having $M(t_0) = 0$ for all t_0.] Third, the ST requires, for a given choice of $\varphi = l_1\pi/l_2$ (see Remark to Lemma II), having a *selective* resonance condition; e.g., for $\varphi = 3\pi/2$, suppression in not predicted for the main resonance ($\Omega = \omega$) for which $\varphi = 2m\pi, m$ an integer.

Computer simulations on the system described by Eq. (1) showed very good agreement between the numerical results and theoretical predictions, even when the perturbation amplitudes and the initial conditions do not fit *sensibly* the MHM requirements. Regular and (transient and

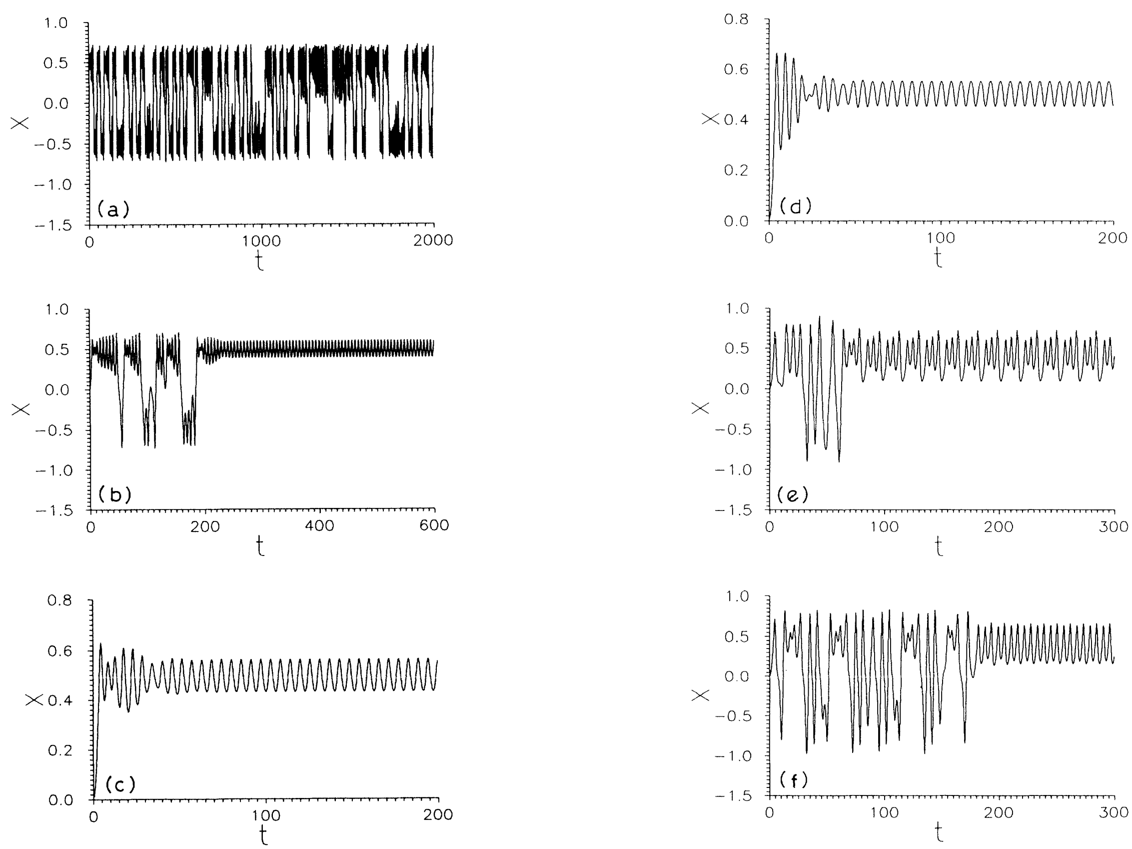

FIG. 3. Displacement time series. The parameters are $\beta=4$, $\delta=0.154$, $\gamma=0.095$, $\omega\equiv\Omega=1.1$, $\varphi=0$. (a) $\eta=0$, (b) $\eta=0.05$, (c) $\eta=\eta_{\min}=0.094\,552\,1$, (d) $\eta=\eta_{\max}=0.264\,097\,5$, (e) $\eta=0.45$, and (f) $\eta=0.5$.

steady) chaotic motions were mainly detected by using time series and power spectra. Note that, in order to obtain precise threshold values of η, Ω, and φ for *complete* regularization, one has to deal with both transient and steady chaos, in addition to periodic motions. This means that the Lyapunov exponent (LE) is not a suitable tool for obtaining such numerical threshold values since it provides information concerning only *steady* motions. In fact, the use of the LE to this end leads to wrong results (as, e.g., in Ref. [5], where the erroneous results are also due to an inadequate choice of φ for the second and third resonances), the correct values for η being lower. Figure 1 shows a characteristic example for a choice of φ and η fitting the ST (really, η in its lower limit) and Ω approaching one of the permitted resonant values ($\Omega = \omega$). Observe the increasing duration of the "laminar" phases and that complete regularization [i.e., when the transient motion is also regular; see Fig. 1(f)] is only achieved at exact resonance. Figure 2 shows the averaged length of the laminar regions $\langle l \rangle$ versus $\varepsilon \equiv \Omega - 1.1$, $\Omega \rightarrow 1.1$. The best fit gives $\langle l \rangle \alpha \varepsilon^{-1.00642}$ in excellent agreement with the prediction of Manneville and Pomeau ($\langle l \rangle \alpha \varepsilon^{-1}$) [15] for type-II and type-III intermittencies. Additional proofs [14] have shown that, in fact, the regularization route is type-II intermittent when Ω approaches (one of) the theoretically predicted value(s).

In order to test the analytical predictions for η, Ω and φ are chosen verifying the ST requirements and then η is varied over the associated allowed interval $]\eta_{\min}, \eta_{\max}]$. Figure 3 shows a typical sequence. In general, the shortest transient to regular motions are found for η values centered in the allowed interval $]\eta_{\min}, \eta_{\max}]$. Finally, Fig. 4 shows a characteristic example testing a theoretical value of φ. (Note that in Ref. [5] it is stated that " ... it is worth mentioning that all the observed phenomenology is independent of the initial phase shift between the two cosines ... .") In this case ($\Omega = 2\omega$) the ST imposes $\varphi = (4m - 1)\pi/2$ m an integer. Observe that the (almost) complete regularization is already achieved for φ very near $3\pi/2$ [Fig. 4(c)] and for $\eta = \eta_{\min}$.

In summary, I have shown that the application of periodic parametric perturbations is an efficient route to reduce or suppress steady chaotic states in nonlinear oscillators. All the observed (numerically and experimentally) phenomenology was theoretically explained by using the MHM for the example of the DH oscillator. A key problem subsists. The effects of such perturbations can be quite difficult to predict, i.e., one generally does not know what type of nonchaotic motion to expect. However, the selective resonance conditions of the ST give us valuable information about its possible Fourier expansions. Such a regular solution should be closely related to some resonant steady periodic solutions of the associated Hamiltonian system. At this stage, however, this connection is speculative.

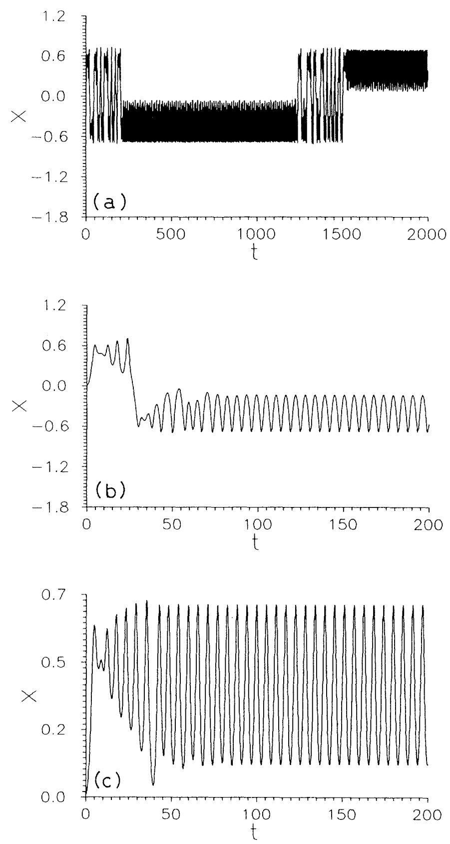

FIG. 4. Displacement time series. The parameters are $\beta = 4$, $\delta = 0.154$, $\gamma = 0.088$, $\omega = 1.1$, $\Omega \equiv 2\omega = 2.2$, $\eta = \eta_{\min} = 0.0642$. (a) $\varphi = 3.141\,592\,7$, (b) $\varphi = 4.0$, and (c) $\varphi = 4.712\,389$.

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